The Representation Theory of Decoherence Functionals in History Quantum Theories[†]

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In the first part of this paper the general perspective of history quantum theories is reviewed. History quantum theories provide a conceptual and mathematical framework for formulating quantum theories without a globally defined Hamiltonian time evolution and for introducing the concept of space-time event into quantum theory. On a mathematical level a history quantum theory is characterized by the space of histories, which represent the space-time events, and by the space of decoherence functionals, which represent the quantum mechanical states in the history approach. The second part of this paper is devoted to the study of the structure of the space of decoherence functionals for some physically reasonable spaces of histories in some detail. The temporal reformulation of standard Hamiltonian quantum theories suggests to consider the case that the space of histories is given by (i) the lattice of projection operators on some Hilbert space or, slightly more generally, (ii) the set of projection operators in some von Neumann algebra. In the case (i) the conditions are identified under which decoherence functionals can be represented by, respectively, trace class operators, bounded operators, or families of trace class operators on the tensor product of the underlying Hilbert space by itself. Moreover, we discuss the naturally arising representations of decoherence functionals as sesquilinear forms. The paper ends with a discussion of the consequences of the results for the general axiomatic framework of history theories.

1. INTRODUCTION

In standard textbook Hamiltonian quantum mechanics the time variable is fixed from the outset as the variable conjugate to the Hamiltonian. A quantum mechanical system is described with the aid of a single-time Hilbert space $\mathfrak{H}_s$. (For simplicity we consider here and in the sequel only quantum systems without superselection rules.) The observables associated with the

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system are identified with self-adjoint operators on the single-time Hilbert space $\mathfrak{H}_s$ and the quantum mechanical states of the physical system with density operators on the single-time Hilbert space $\mathfrak{H}_s$. As is obvious from these statements, all observables and states are associated with a fixed time (or slightly more generally with a fixed spacelike hypersurface) and there is no notion of observable associated with an extended region of space-time in standard quantum mechanics. The time evolution is governed by certain unitary operators on the single-time Hilbert space $\mathfrak{H}_s$.

The aim of the history approach (at least for the purpose of the present investigation) is to formulate an intrinsically quantum mechanical formalism in which observables and states are associated with extended space-time regions and in which time plays a potentially subsidiary role. There have been a few attempts in the literature to extrapolate the usual Hilbert space formalism to situations involving observables associated with extended space-time regions in an *ad hoc* way. However, it is not *a priori* clear whether such a simple strategy can be justified. In contrast in the history approach [particularly in the approach pioneered by Isham (1994)], one proceeds along a different route. Methodologically, what one is trying to do is to find a quantum mechanical formalism involving space-time observables and states by starting by suitably reformulating standard quantum mechanics. In the present paper we review the progress of this program which has been made in the recent years, (Isham, 1994, 1997; Isham and Linden, 1994, 1995 1997; Isham *et al.*, 1994, 1998; Rudolph, 1996a, 1996b, 1998, 1999a, 1999b), Rudolph and Wright (1997, 1998, 1999), Schreckenberg, 1995, 1996a, b, 1997; Wright, 1995, 1998, 1999a, 1999b).

This article is structured as follows. In the first part (Sections 2 and 3) we review the general framework and perspective of so-called history quantum theories. Specifically, we discuss the history reformulation of nonrelativistic quantum mechanics in the case that the underlying single-time Hilbert space $\mathfrak{H}_s$ is finite dimensional and introduce the notion of decoherence functional, which represent the states in the present approach. This history reformulation of standard quantum mechanics serves as motivation for Isham's algebraic axiomatization of general history quantum theories. The temporal reformulation of standard quantum mechanics suggests we consider general history quantum theories for which the space of histories is given by (i) the lattice of projection operators on some Hilbert space or, slightly more generally, (ii) the set of projection operators in some von Neumann algebra. The second main part of this article (Section 4) is devoted to the representation theory of decoherence functionals both in general history theories and in the history reformulation of standard quantum mechanics. If the underlying history Hilbert space $\mathscr{H}$ is finite dimensional, a complete classification of decoherence functionals has been given by Isham, *et al.* (1994) with their so-called ILS-

theorem, which establishes a one-to-one correspondence between bounded decoherence functionals and certain trace class operators. In the case that the history Hilbert space $\mathcal{H}$ is infinite dimensional, we shall be concerned with the problem of determining the decoherence functionals for which the ILS-theorem can be generalized. Moreover, we shall discuss the natural representations of bounded decoherence functionals as sesquilinear forms with a natural representation on a Hilbert space. Since the history reformulation of standard nonrelativistic quantum mechanics is the motivating example for the history approach, we shall be particularly interested in representations of the standard decoherence functional d_ρ associated with the initial state ρ in standard quantum mechanics. We conclude with a discussion of our results for the general framework for history theories proposed by Isham and put forward a modified axiomatization of history quantum theories.

Notations and Conventions

Throughout this work we will make use of Dirac's ket and bra notation to denote vectors in Hilbert space and dual vectors in the dual Hilbert space, respectively. We adopt the convention that inner products of Hilbert spaces are linear in the second variable and conjugate linear in the first variable.

Throughout this work $\mathfrak{H}$ and $\mathcal{H}$ denote Hilbert spaces, $\mathcal{P}(\mathfrak{H})$ denotes the lattice of all projection operators on a Hilbert space $\mathfrak{H}$, $\mathcal{B}(\mathfrak{H})$ denotes the set of all bounded operators on $\mathfrak{H}$, and $\mathcal{K}(\mathfrak{H})$ denotes the set of compact operators on $\mathfrak{H}$. The tensor product of the two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is denoted by $\mathcal{H}_1 \otimes \mathcal{H}_2$. The algebraic tensor product of $\mathcal{K}(\mathcal{H}_1)$ with $\mathcal{K}(\mathcal{H}_2)$ is denoted by $\mathcal{K}(\mathcal{H}_1) \otimes_{alg} \mathcal{K}(\mathcal{H}_2)$. The symbol $\mathfrak{H}_s$ always denotes the single-time Hilbert space in standard quantum mechanics.

2. THE HISTORY REFORMULATION OF STANDARD QUANTUM MECHANICS

2.1. Homogeneous Histories

Our starting point toward a formal definition of the notion of history is the observation that, by virtue of the spectral theorem, every observable in standard quantum mechanics can be disintegrated in two-valued yes–no observables which are represented by projection operators on the single-time Hilbert space $\mathfrak{H}_s$. In a first step toward the history reformulation of standard quantum mechanics one considers finite sequences of projection operators (Griffiths, 1984)

$$h = P_{t_1}, P_{t_2}, \ldots, P_{t_n}$$

labeled by a discrete set of time parameters $\{t_1, \ldots, t_n\}$. We call such a

sequence h a *homogeneous history*. Operationally, one may think of such a sequence as representing a sequence of possible measurement outcomes.

Standard quantum mechanics suggests the following Ansatz for the quantum mechanical probability of a homogeneous history h in the state ρ, which we denote by the symbol $d_\rho(h, h)$:

$$d_\rho(h, h) = \text{tr}_{\mathfrak{H}_s}(P_{t_n} \cdots P_{t_2} P_{t_1} \rho P_{t_1} P_{t_2} \cdots P_{t_n}). \tag{1}$$

This expression coincides with the formula for the probability of the sequence of measurement outcomes corresponding to the sequence of projections $\{P_{t_1}, \ldots, P_{t_n}\}$ in a measurement situation (Wigner, 1963). [Notice that we are working in the Heisenberg picture here and suppress for notational simplicity the unitary time evolution operators in the expression for the probability.]

At this stage we face a list of problems:

1. The space of all homogeneous histories carries no obvious "nice" and simple mathematical structure and, particularly, in general it is not obvious what are the appropriate mathematical representatives corresponding to propositions like, "*the history h or the history k is realized*," "*the histories h and k are both realized*," and "*the history h is not realized*."
2. There is no notion of "sum" of homogeneous histories.
3. Therefore there is no additive probability measure on the space of homogeneous histories.

2.2. Temporal Quantum Logic

The first two problems have been solved by Isham (1994). He observed that every homogeneous history $h = \{h_{t_i}\}$ can be canonically mapped to some projection operator on the *n*-fold tensor product Hilbert space $\otimes_{t_i} \mathfrak{H}_{t_i}$ (where $\mathfrak{H}_{t_i} = \mathfrak{H}_s$ for all i) of the single time Hilbert space $\mathfrak{H}_s$ by itself via

$$h \simeq \{h_{t_i}\} \mapsto h_{t_1} \otimes \cdots \otimes h_{t_n}.$$

Now we observe that the space $\mathscr{P}(\otimes_{t_i} \mathfrak{H}_{t_i})$ of all projections on this tensor product Hilbert space carries the structure of a lattice. The central postulate in Isham's temporal quantum logic is to identify all projections in $\mathscr{P}(\otimes_{t_i} \mathfrak{H}_{t_i})$ with physical histories. The lattice-theoretic operations in $\mathscr{P}(\otimes_{t_i} \mathfrak{H}_{t_i})$ then provide a natural solution to the first two problems mentioned above.

The space of all histories can then be identified with the following direct limit:

$$\mathcal{P} := \lim\{\mathcal{P}(\otimes_{t_i \in l} \mathfrak{H}_{t_i}) | l \subset \mathbb{R} \text{ finite}\}.$$

All histories in $\mathcal{P}$ which are not homogeneous are also called *inhomogeneous histories*.

Having identified the space of histories in the history reformulation of standard quantum mechanics, we are now interested in knowing what the dual notion representing the states is.

2.3. Decoherence Functionals

A *decoherence functional d* is a bivariate, complex-valued functional $d: \mathcal{P} \times \mathcal{P} \to \mathbb{C}$ such that for all $\alpha, \alpha', \beta \in \mathcal{P}$ with $\alpha \perp \alpha'$:

- $d(\alpha, \alpha) \in \mathbb{R}$ and $d(\alpha, \alpha) \geq 0$.
- $d(\alpha, \beta) = d(\beta, \alpha)^*$.
- $d(1, 1) = 1$ and $d(0, \alpha) = 0$ for all α.
- $d(\alpha \vee \alpha', \beta) = d(\alpha, \beta) + d(\alpha', \beta)$.

The idea behind the positivity requirement for the diagonal values of d is that $d(\alpha, \alpha)$ represents the probability of the history α.

The prime and motivating example for a decoherence functional with the above list of properties is the decoherence functional d_ρ in standard quantum mechanics associated with the initial state ρ, which is defined for homogeneous histories $h \simeq \{h_{t_i}\}$ and $k \simeq \{k_{t_j}\}$ by

$$d_\rho(h, k) := \mathrm{tr}(h_{t_n} h_{t_{n-1}} \cdots h_{t_1} \rho k_{t_1} \cdots k_{t_n}). \tag{2}$$

This is a modest generalization of the above expression for the probability $d_\rho(h, h)$ of some history h. When the single-time Hilbert space $\mathfrak{H}_s$ is finite dimensional, then the decoherence functional d_ρ so defined can uniquely be extended to a bi-additive function on the set of all histories $\mathcal{P}$, as will be shown below. However, we shall also argue below that d_ρ cannot be extended to a finitely valued functional on the set $\mathcal{P}$ of all histories if the single-time Hilbert space is infinite dimensional.

2.4. ILS-Representation for d_ρ

The decoherence functional d_ρ associated with the initial state ρ in standard quantum mechanics defined for homogeneous histories by Equation (2) admits a so-called Isham–Linden–Schreckenberg representation (ILS-representation) (Isham *et al.*, 1994). This means that for all *n*-time histories $h = h_{t_1} \otimes \cdots \otimes h_{t_n}$ and $k = k_{t_1} \otimes \cdots \otimes k_{t_n}$ there exists a trace class operator $\mathfrak{X}_\rho$ on the 2*n*-fold tensor product $\mathcal{H}_n \otimes \mathcal{H}_n = \mathfrak{H}_{t_1} \otimes \cdots \otimes \mathfrak{H}_{t_n} \otimes \mathfrak{H}_{t_1} \otimes \cdots \otimes \mathfrak{H}_{t_n}$ of the single-time Hilbert space $\mathfrak{H}_s$ by itself such that d_ρ can be represented as

$$d_\rho(h, k) = \text{tr}_{\mathcal{H}_n \otimes \mathcal{H}_n}(h \otimes k\mathfrak{X}_\rho). \tag{3}$$

The operator $\mathfrak{X}_\rho$ depends both on the initial state ρ and on the unitary time evolution operator $U(t, t')$. The dependence on U can explicitly be split up as

$$\mathfrak{X}_\rho = (U^\dagger_{t_1,t_2,...,t_n} \otimes U^\dagger_{t_1,t_2,...,t_n})\mathfrak{Y}_\rho(U_{t_1,t_2,...,t_n} \otimes U_{t_1,t_2,...,t_n}))$$

where

$$U_{t_1,t_2,...,t_n} := U(t_0, t_1) \otimes U(t_0, t_2) \otimes \cdots \otimes U(t_0, t_n).$$

We are left with an operator $\mathfrak{Y}_\rho$ depending only on the initial state ρ. The operator $\mathfrak{Y}_\rho$ admits a representation as a series

$$\mathfrak{Y}_\rho = \sum_{i_1,...,i_{2n}} \omega_{i_1}\{|e^1_{i_1}\rangle\langle e^{2n}_{i_{2n}}| \otimes |e^{2n}_{i_{2n}}\rangle\langle e^{2n-1}_{i_{2n-1}}| \otimes \cdots \otimes |e^{n+2}_{i_{n+2}}\rangle\langle e^{n+1}_{i_{n+1}}|$$

$$\otimes |e^2_{i_2}\rangle\langle e^1_{i_1}| \otimes |e^3_{i_3}\rangle\langle e^2_{i_2}] \otimes \cdots \otimes |e^{n+1}_{i_{n+1}}\rangle\langle e^n_{i_n}|\}.$$

The ω_i are determined by the spectral resolution of $\rho = \Sigma_i \, \omega_i|e^\rho_i\rangle\langle e^\rho_i|$. The orthonormal bases $\{|e^i_{i_j}\rangle\}$, $j \in \{2, \ldots, 2n\}$, are completely arbitrary, whereas $|e^1_i\rangle = |e^\rho_i\rangle$ for all i.

If we restrict ourselves to homogeneous histories h and k, then the ILS-representation in Equation (3) is valid both when the single-time Hilbert space is finite dimensional and when it is infinite dimensional. In the finite-dimensional case $\mathfrak{X}_\rho$ is a trace class operator. Thus, trivially, d_ρ can be extended to the set $\mathcal{P}$ of all histories. In the infinite-dimensional case it can be shown (Rudolph and Wright, 1999) that $\mathfrak{X}_\rho$ is only a bounded operator. Therefore in the infinite-dimensional case d_ρ can in general not be extended to the space of all histories. We shall come back to this issue below.

2.5. Consistent Sets of Histories

It remains to solve the third problem mentioned in Section 2.1, that there is no additive probability measure on the space of homogeneous histories. We have seen that in the history approach the states are identified with decoherence functionals. Again, these decoherence functionals do not define a probability measure on the set of all histories.

The situation is analogous to standard single-time quantum mechanics. Here the states are given by density operators on the single-time Hilbert space which do not induce probability measures on the set of *all* observables. In standard quantum mechanics we call a set of observables *compatible* if the state induces a joint probability measure on the set of possible values of the observables. It is well known that a set of observables is compatible if and only if the associated self-adjoint operators are pairwise commuting

(particularly, the notion of compatibility of observables is independent of the state).

Generalizing this point of view to the histories approach, one calls a set of histories *consistent* if the decoherence functional induces a probability measure on this set of histories. It is easy to prove that a Boolean sublattice $\mathscr{C}$ of $\mathscr{P}$ is consistent if and only if $\text{Re } d_\rho(h, k) = 0$ for all orthogonal $h, k \in \mathscr{C}$. The consistent sets of histories are thus the generalizations of commuting, compatible observables in standard quantum mechanics, and the existence of several mutually inconsistent consistent sets is just the expression of the complementarity principle in the histories approach.

3. GENERAL HISTORY THEORIES

The history reformulation of standard quantum mechanics in finite dimensions reviewed above led Isham (1994) to his axiomatic framework for general history quantum theories. According to his framework, a general history theory is characterized by two sets.

First there is the space of histories $\mathscr{U}$, which carries the structure of a lattice, an orthoalgebra, a D-poset, or another algebraic structure such that (i) there is a partial order defined, (ii) there is a least element 0 and a greatest element 1 with respect to this partial order, and (iii) there is a notion of orthogonality between elements (denoted by $\perp$) and a notion of sum (denoted by $\oplus$) for orthogonal elements. The tentative physical interpretation of the histories is that they represent propositions about events in extended regions of space-time.

Dual to the space of histories is the space of decoherence functionals, which represent the generalized states in the history approach. Recall that decoherence functionals are bivariate, complex-valued functionals $d: \mathscr{U} \times \mathscr{U} \to \mathbb{C}$ such that for all $\alpha, \alpha', \beta \in \mathscr{U}$ with $\alpha \perp \alpha'$:

- $d(\alpha, \alpha) \in \mathbb{R}$ and $d(\alpha, \alpha) \geq 0$.
- $d(\alpha, \beta) = d(\beta, \alpha)^*$.
- $d(1, 1) = 1$ and $d(0, \alpha) = 0$ for all α.
- $d(\alpha \oplus \alpha', \beta) = d(\alpha, \beta) + d(\alpha', \beta)$.

The history reformulation of standard quantum mechanics in finite dimensions suggests that a natural choice for the space of histories is given by the set of projection operators on some (history) Hilbert space $\mathscr{H}$ or in some von Neumann algebra $\mathscr{A}$. In the rest of this paper we shall exclusively consider these two cases. The second part of this paper is devoted to the problem of determining what can be said about the structure of the space of decoherence functionals for these choices of the space of histories.

4. THE REPRESENTATION THEORY OF DECOHERENCE FUNCTIONALS

4.1. Finite-Dimensional History Hilbert Spaces

First let us consider an abstract history theory (as described in Section 2.5) for which the space of histories is given by the set of projection operators $\mathcal{P}(\mathcal{H})$ on some finite-dimensional Hilbert space $\mathcal{H}$ (with dimension greater than two).

In this case the classification problem for decoherence functionals has been completely solved by Isham *et al.* (1994). According to their result, there exists a one-to-one correspondence between uniformly bicontinuous decoherence functionals d for $\mathcal{H}$ and trace class operators $\mathfrak{X}$ on $\mathcal{H} \otimes \mathcal{H}$ according to the rule

$$d(p, q) = \text{tr}_{\mathcal{H} \otimes \mathcal{H}}((p \otimes q)\mathfrak{X}) \qquad (4)$$

for all projections $p, q \in \mathcal{P}(\mathcal{H})$ with the following restrictions:

- $\text{tr}_{\mathcal{H} \otimes \mathcal{H}}((p \otimes q)\mathfrak{X}) = \text{tr}_{\mathcal{H} \otimes \mathcal{H}}((q \otimes p)\mathfrak{X}^*)$.
- $\text{tr}_{\mathcal{H} \otimes \mathcal{H}}((p \otimes p)\mathfrak{X}) \geq 0$.
- $\text{tr}_{\mathcal{H} \otimes \mathcal{H}}(\mathfrak{X}) = 1$.

In particular, every such decoherence functional is bounded. This is result is often also referred to as the Isham–Linden–Schreckenberg theorem (ILS-theorem).

4.2. Infinite-Dimensional History Hilbert Spaces

4.2.1. ILS-Type Representations

A question which arises immediately is, What can be said for history theories where the space of histories is given by the set of projections on some infinite-dimensional Hilbert space $\mathcal{H}$?

In the sequel we will make use of the following theorem, which is a special case of a more general result proved in Wright (1995).

Theorem 4.1. Let $\mathcal{H}$ be a Hilbert space which is either infinite dimensional or of finite dimension greater than two. Then a decoherence functional d *on* $\mathcal{H}$ can be extended (uniquely) to a bounded bilinear form $\mathcal{D}: \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{C}$ if, and only if, d is bounded.

An immediate consequence of Theorem 4.1 is then that, by the fundamental property of the algebraic tensor product, there is a unique linear functional $\beta: \mathcal{K}(\mathcal{H}) \otimes_{\text{alg}} \mathcal{K}(\mathcal{H}) \rightarrow \mathbb{C}$ on the algebraic tensor product of $\mathcal{K}(\mathcal{H})$ by itself such that

$$\beta(x \otimes y) = \mathcal{D}(x, y) \tag{5}$$

for all $x, y \in \mathcal{K}(\mathcal{H})$. In particular, $d(p, q) = \beta(p \otimes q)$ for all projections p and q in $\mathcal{K}(\mathcal{H})$.

The functional β can now be used to completely characterize the set of decoherence functionals in the infinite-dimensional case admitting an ILS-representation.

Tensor Bounded Decoherence Functionals

Definition. A decoherence functional d is said to be *tensor bounded* if the associated functional β is bounded on $\mathcal{K}(\mathcal{H}) \otimes_{\mathrm{alg}} \mathcal{K}(\mathcal{H})$, when $\mathcal{K}(\mathcal{H}) \otimes_{\mathrm{alg}} \mathcal{K}(\mathcal{H})$ is equipped with its unique pre-C^*-norm induced by the operator norm on $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$.

Theorem 4.2. Let $\mathcal{H}$ be a Hilbert space which is not of dimension two. Let d be a bounded decoherence functional for $\mathcal{H}$. Then d is tensor bounded if, and only if, there exists a trace class operator $\mathfrak{X}$ on $\mathcal{H} \otimes \mathcal{H}$ such that

$$d(p, q) = \mathrm{tr}_{\mathcal{H} \otimes \mathcal{H}}((p \otimes q)\mathfrak{X}) \tag{6}$$

for all projections p and q in $\mathcal{P}(\mathcal{H})$.

Thus we conclude that in the infinite-dimensional case a bounded decoherence functional d admits an ILS-representation if and only if it is tensor bounded.

Tracially Bounded Decoherence Functionals. There is another physically important class of decoherence functionals, the so-called tracially bounded decoherence functionals.

Definition. A decoherence functional d is said to be *tracially bounded* if it is bounded and, when β is the corresponding linear functional on $\mathcal{K}(\mathcal{H}) \otimes_{\mathrm{alg}} \mathcal{K}(\mathcal{H})$, there exists a constant C such that, for each unit vector ξ in $\mathcal{H} \otimes_{\mathrm{alg}} \mathcal{H}$, $|\beta(|\xi\rangle\langle\xi|)| \leq C$.

Then we have the following theorem.

Theorem 4.3. Let the decoherence functional d be tracially bounded for $\mathcal{H}$, where $\mathcal{H}$ is separable and of dimension greater than two. Then there exists a unique bounded linear operator $\mathfrak{M}$ on $\mathcal{H} \otimes \mathcal{H}$ such that

$$d(p, q) = \mathrm{tr}_{\mathcal{H} \otimes \mathcal{H}}(\mathfrak{M}(p \otimes q)) \tag{7}$$

whenever p and q are finite-rank projections on $\mathcal{H}$. Let d be moreover countably additive, then whenever p and q are projections in $\mathcal{P}(\mathcal{H})$ and $\{p_n\}_{n \in \mathbb{N}}$ and $\{q_n\}_{n \in \mathbb{N}}$ are, respectively, orthogonal families of finite-rank projections with $p = \Sigma_{n \in \mathbb{N}} p_n$ and $q = \Sigma_{n \in \mathbb{N}} q_n$,

$$d(p, q) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mathrm{tr}_{\mathcal{H} \otimes \mathcal{H}}((p_i \otimes q_j)\mathfrak{M}). \qquad (8)$$

Thus we see that tracially bounded decoherence functionals admit a pseudo-ILS-representation by some bounded operator as in Equation (7).

4.2.2. The Standard Decoherence Functional d_ρ in Infinite Dimensions

Tracially bounded decoherence functionals are of particular interest since the standard decoherence functional d_ρ in the history reformulation of standard quantum mechanics is tracially bounded.

The proof is quite easy (Rudolph and Wright 1999). In finite dimensions this is trivial. In infinite dimensions the argument is as follows. For simplicity of notation we consider only two time histories; the general case is analogous. First recall the ILS-representation of d_ρ valid for pairs of homogeneous histories p, q,

$$d_\rho(p, q) = \sum_{j_1,\ldots,j_4=1}^{\dim \mathfrak{H}_s} \omega_{j_1} \langle e_{j_4}^4 \otimes e_{j_3}^3 \otimes \psi_{j_1} \otimes e_{j_2}^2,$$

$$(p \otimes q)(\psi_{j_1} \otimes e_{j_4}^4 \otimes e_{j_2}^2 \otimes e_{j_3}^3)\rangle.$$

It is easy to see that the series still converges if we replace $p \otimes q$ by a compact operator of rank one. This implies that we can define a sesquilinear form S_ρ by

$$S_\rho(\xi, \eta) = \sum_{j_1,\ldots,j_4=1}^{\dim \mathfrak{H}_s} \omega_{j_1} \langle e_{j_4}^4 \otimes e_{j_3}^3 \otimes \psi_{j_1} \otimes e_{j_2}^2, \xi \rangle \langle \eta, \psi_{j_1} \otimes e_{j_4}^4 \otimes e_{j_2}^2 \otimes e_{j_3}^3 \rangle.$$

The Cauchy–Schwarz inequality implies that S_ρ is bounded: $|S_\rho(\xi, \eta)| \leq \|\xi\| \cdot \|\eta\|$, which in turn implies $|\beta_\rho(p_\xi)| = |S_\rho(\xi, \xi)| \leq 1$. This implies that there exists a bounded operator $\mathfrak{X}_\rho$ such that $S_\rho(\xi, \eta) = \langle \eta, \mathfrak{X}_\rho \xi \rangle$ and by straightforward computation one verifies that $\mathfrak{X}_\rho$ coincides with the ILS-operator associated with d_ρ in Section 2.4. Thus d_ρ admits a pseudo-ILS-representation as in Theorem 4.3 with $\mathfrak{M}$ replaced by $\mathfrak{X}_\rho$ and since d_ρ is, moreover, countably additive, the analogue of Equation (8) is also satisfied (whenever well defined).

As we shall see below, d_ρ is in general not bounded (and not even finitely valued) on the space of all histories $\mathcal{P}$. Thus we cannot apply Theorem 4.1 to infer the existence of the functional β_ρ associated with d_ρ. However, from the ILS-series for d_ρ we can directly infer the existence of β_ρ on a suitably smaller chosen domain of definition.

Nonexistence of a Finitely Valued Extension of d_ρ. As already mentioned repeatedly, if the single-time Hilbert space is infinite dimensional, then the

standard decoherence functional d_ρ defined on homogeneous histories by Equation (2) cannot be extended to a finitely valued functional on the set of all projection operators on the tensor product Hilbert space. We assume for simplicity that the single-time Hilbert space is separable. Consider the ILS-representation for d_ρ in Equation (3). For simplicity of notation we consider the case $n = 2$. We define

$$D_\rho(p, q) = \sum_{j_1,\ldots,j_4=1}^{\dim \mathfrak{H}} \omega_{j_1} \langle e_{j_4}^4 \otimes e_{j_3}^3 \otimes \psi_{j_1} \otimes e_{j_2}^2,$$

$$(p \otimes q)(\psi_{j_1} \otimes e_{j_4}^4 \otimes e_{j_2}^2 \otimes e_{j_3}^3) \rangle \tag{9}$$

for all histories $p, q \in \mathcal{P}(\mathfrak{H}_{t_1} \otimes \mathfrak{H}_{t_2})$ for which the sum converges. Now choose $e_j^4 = e_j^3 = e_j^2 = \psi_j$ for all j. Fix i_1 and let $\varphi_i := (1/\sqrt{2})(|\psi_i \otimes \psi_{i_1}\rangle + |\psi_{i_1} \otimes \psi_i\rangle)$ for every $i \in \mathbb{N} \setminus \{i_1\}$. Then clearly $\varphi_i \perp \varphi_j$ if $i \neq j$. Set $f_{j_1,j_2,j_3}(q) := \langle \psi_{j_1} \otimes \psi_{j_2}, q(\psi_{j_2} \otimes \psi_{j_3}) \rangle$, then an easy computation shows that

$$D_\rho(P_{\varphi_i}, q) = \frac{1}{2} \sum_{j_2} (\omega_{i_1} f_{i_1,j_2,i_1}(q) + \omega_i f_{i,j_2,i}(q))$$

for $i \neq i_1$, where P_{φ_i} denotes the projection operator onto the subspace spanned by φ_i. Put $P = \sum_{i \neq i_1} P_{\varphi_i}$; then clearly the expression in Equation (9) for $D_\rho(P, q)$ does not converge for arbitrary q.

This proves that if the single-time Hilbert space is infinite dimensional, there does not exist a finitely valued extension of d_ρ to the set of histories $\mathcal{P}$.

4.2.3. Bounded Decoherence Functionals

We now return to our discussion of representations of decoherence functionals in general history theories. We have identified above the classes of decoherence functionals admitting an ILS-representation and a pseudo-ILS-representation, respectively. It is also of some interest to see what can be said about general bounded decoherence functionals. Although a bounded decoherence functional in general does not admit an ILS-representation, they can be approximated by a series of ILS-representable decoherence functionals in the following sense.

Proposition 4.4. Let $\mathcal{H}$ be a Hilbert space with $\dim(\mathcal{H}) > 2$ and let d: $\mathcal{P}(\mathcal{H}) \times \mathcal{P}(\mathcal{H}) \to \mathbb{C}$ be a bounded decoherence functional for $\mathcal{H}$. Then there exist families of trace class operators $\{\mathcal{X}_i\}_{i \in I}$ and $\{\mathcal{Y}_i\}_{i \in I}$ on $\mathcal{H}$, where, for each x and y in $\mathcal{K}(\mathcal{H})$, $\sum_{i \in I} |\mathrm{tr}_\mathcal{H}(x\mathcal{X}_i)|^2$ and $\sum_{i \in I} |\mathrm{tr}_\mathcal{H}(y\mathcal{Y}_i)|^2$ are convergent and, for all $p, q \in \mathcal{K}(\mathcal{H})$,

$$d(p, q) = \sum_{i \in I} \mathrm{tr}_{\mathcal{H} \otimes \mathcal{H}}(p \otimes q(\mathcal{X}_i \otimes \mathcal{X}_i^* - \mathcal{Y}_i \otimes \mathcal{Y}_i^*)) \tag{10}$$

where the infinite series is absolutely convergent.

4.3. Representations as Sesquilinear Forms

4.3.1. Bounded Decoherence Functionals

There is an alternative representation theorem for bounded decoherence functionals in general history theories as bounded sesquilinear forms on a Hilbert space due to Wright (1995). This is valid also for history theories over von Neumann algebras (with no type I_2 direct summand).

Theorem 4.5. Let A be a von Neumann algebra with no type I_2 direct summand and $d\colon \mathcal{P}(A) \times \mathcal{P}(A) \to \mathbb{C}$ a bounded decoherence functional. Then there exists a map $x \mapsto [x]$ from A into a dense subspace of a Hilbert space $\mathcal{H}$ and a self-adjoint operator T on $\mathcal{H}$ such that

$$D(x, y) = \langle T[x], [y] \rangle$$

is an extension of d.

Alternatively, there exist semi-inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ on $\mathcal{H}$ such that

$$d(p, q) = \langle p, q \rangle_1 - \langle p, q \rangle_2.$$

The proof makes use of the profound Haagerup–Pisier–Grothendieck inequality to associate a state (in the C^*-algebraic sense) with the decoherence functional. The Hilbert space is then constructed via a GNS-type construction.

4.3.2. Standard Decoherence Functional

As shown above, in standard quantum mechanics the standard decoherence functional d_ρ does not admit a finitely valued extension to the set of all histories in Isham's framework. Thus there is also no hope to represent it as a bounded sesquilinear form on some Hilbert space. However, there is a natural representation of the standard decoherence functional as an unbounded sesquilinear form, which in brief can be constructed as follows.

If $\mathfrak{H}_s$ is infinite dimensional, then d_ρ can be extended to a bilinear functional on $\mathcal{B}(\mathfrak{H}_s) \otimes_{\mathrm{alg}} \cdots \otimes_{\mathrm{alg}} \mathcal{B}(\mathfrak{H}_s)$ (n times) as

$$D_\rho(b, b') := \mathrm{tr}(\Pi(b')^\dagger \Pi(b)\rho)$$

where Π is defined on homogeneous elements by $\Pi(b_1 \otimes \cdots \otimes b_n) = b_n \cdots b_1$ and extended to all of $\mathcal{B}(\mathfrak{H}_s) \otimes_{\mathrm{alg}} \cdots \otimes_{\mathrm{alg}} \mathcal{B}(\mathfrak{H}_s)$ by linearity.

Theorem 4.6. There exists a Hilbert space $\mathcal{H}$ and a linear operator R_ρ from $\mathcal{B}(\mathfrak{H}_s) \otimes_{\mathrm{alg}} \cdots \otimes_{\mathrm{alg}} \mathcal{B}(\mathfrak{H}_s)$ into a dense subspace of $\mathcal{H}$ such that

$$D_\rho(b, b') = \langle R_\rho(b'), R_\rho(b) \rangle$$

for all $b, b', \in \mathcal{B}(\mathfrak{H}_s) \otimes_{\mathrm{alg}} \cdots \otimes_{\mathrm{alg}} \mathcal{B}(\mathfrak{H}_s)$. (Here $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathcal{H}$.) R_ρ is unbounded if and only if $\mathfrak{H}_s$ is infinite dimensional.

5. GENERAL HISTORY THEORIES II

At this stage it is worthwhile to recall that Isham's axiomatic framework for general history theories sketched in Section 3 was motivated by the history reformulation of standard quantum mechanics in *finite* dimensions.

The results reported in this paper, in particular the negative result that the standard decoherence functional in infinite dimensions cannot be extended to the space of "all" histories in Isham's framework on the one hand and the positive result that the standard decoherence functional admits nevertheless a natural representation as an unbounded sesquilinear form on some Hilbert space on the other hand, indicate that Isham's axiomatic framework needs to be modified.

We conclude this paper by indicating what in our opinion is the appropriate structure.

According to our proposal a general history theory is characterized by two sets.

1. The set of propositions which is embedded into a Hilbert space $\mathcal{H}$. The propositions are interpreted in physical terms as propositions about events in extended regions of space-time.
2. The set of states which are identified with bounded or unbounded sesquilinear forms s on the Hilbert space $\mathcal{H}$.

The probability of a proposition $x \in \mathcal{D}(s)$ in the domain of definition of some sesquilinear form is given by $s(x, x)$. This framework for temporal quantum theories has been discussed in more detail in Rudolph (1999a).

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